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The Poisson–Lie structure of nonlinear O(N) σ-model by using the moving-frame method

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Abstract. We discuss the Poisson–Lie structure of the integrable nonlinear O(N) σ-model with the moving-frame method. The corresponding r- and s-matrices are given explicitly. We also perform the gauge transformation for the Lax potential and the r and s matrices. Furthermore, we discover that the field-dependent terms in our r- and s-matrices only depend on the Riemannian connection of the target manifold.

1. Introduction

Great progress has been made in understanding the algebraic structures of two-dimensional nonlinear integrable models with the Hamiltonian approach. The starting point of the discussion is to study the Poisson bracket between Lax potentials. For a lot of integrable models, such as the WZNW models and Toda systems, this bracket leads to a Lie–Poisson algebra as [5]

$$\{L(x,\lambda)\underset{,}{\otimes}L(y,\mu)\}=[r(\lambda,\mu),L(x,\lambda)\otimes 1+1\otimes L(x,\mu)]\delta(x-y). \qquad (1)$$

with an antisymmetric r-matrix acting as its structural constant. This matrix, known as the classical r-matrix, satisfies the famous classical Yang–Baxter equation

$$[r_{12}(\lambda,\mu),r_{13}(\lambda,\nu)]+[r_{12}(\lambda,\mu),r_{23}(\mu,\nu)]+[r_{13}(\lambda,\nu),r_{23}(\mu,\nu)]=0 \qquad (2)$$

so that the Poisson structure of the dynamical systems is consistent. The importance of structure (1) lies in the central role it plays in the context of integrable systems [5]. The models fitting equation (1) are called ultralocal because the RHS of equation (1) contains only the delta function $\delta(x-y)$ but not its derivatives. An important generalization of the above Lie–Poisson structure to certain non-ultralocal models has been developed by Maillet [1]. In his new integrable canonical structure, equation (1) is replaced by

$$\begin{aligned}\{L(x,\lambda)\underset{,}{\otimes}L(y,\mu)\}=&-[r(x,\lambda,\mu),L(x,\lambda)\otimes 1+1\otimes L(x,\mu)]\delta(x-y)\\&+[s(x,\lambda,\mu),L(x,\lambda)\otimes 1-1\otimes L(x,\mu)]\delta(x-y)\\&-(r(x,\lambda,\mu)+s(x,\lambda,\mu)-r(y,\lambda,\mu)+s(y,\lambda,\mu))\delta'(x-y).\end{aligned} \qquad (3)$$

Besides the usual antisymmetric r-matrix, another symmetric s structural matrix is introduced in the new structure, and they both generally depend on the fields of the theory. This algebraic structure is the extension of the usual Lie–Poisson algebra for non-ultralocal integrable systems such as the nonlinear integrable σ-models and principal chiral models, and plays a prominent role in them.

Integrable nonlinear σ-models have clear geometric structures: their target manifolds are Riemannian symmetric spaces. Recently, Forger *et al* obtained a pair of field-dependent r- and s-matrices of the σ-models defined on Riemannian symmetric spaces [2]. However, due to the special geometric structure of the models, we still expect that r- and s-matrices have some geometrical meaning. Since geometric structure might be seen more clearly under transformations, we study the O(N) σ-model with a different method—the so-called moving frame method. This method allows us to take gauge transformations for Lax matrices and r- and s-matrices conveniently. By using this method, we get a different form of the r- and s-matrices whose field-dependent terms are, as we expect, just the Riemannian connections on an $(N-1)$-dimensional sphere S^{N-1}, the target manifold of the O(N) σ-model. Furthermore, we find that the new form of r- and s-matrices can be changed into the form obtained by Forger *et al* after a special gauge transformation. Here we note that the discussion can be generalized to any Riemannian symmetric space. A paper is being prepared on this.

This paper is arranged as follows. In section 2, we review some important aspects of the O(N) σ-model and give a new form of Lax pairs in moving frames. In section 3, we work out the new form of r- and s-matrices under the simplest gauge. On the basis of the results obtained in section 2, we get the r- and s-matrices under any gauge in section 4. These results show that the field-dependent terms of the r- and s-matrices are Riemannian connections.

2. O(N) σ-model

A two-dimensional nonlinear σ-model is a field theory in two-dimensional Minkovski space. Its Lagrangian is

$$\mathscr{L} = \tfrac{1}{2} g_{ij}\, \partial_\mu u^i\, \partial^\mu u^j \tag{4}$$

where u^i's are the local coordinates of the target manifold of the model and $\{g^{ij}\}$ is its Riemannian metric matrix. For the O(N) σ-model, its target manifold $S^{N-1} \sim SO(N)/SO(N-1)$ is a Riemannian symmetric space, so there exists an involution operator $n(n^2=1$, but $n \neq 1)$. By using it, the Lie algebra $\mathscr{G}$ of $SO(N)$ can be decomposed as

$$\mathscr{G} = \mathscr{H} + \mathscr{K} \tag{5}$$

$$[n, \mathscr{K}] = 0 \qquad [n, \mathscr{K}]_+ \equiv n\mathscr{K} + \mathscr{K} n = 0$$

so that $\mathscr{H}$ and $\mathscr{K}$ satisfy the following relations:

$$[\mathscr{H}, \mathscr{H}] \subset \mathscr{H} \qquad [\mathscr{H}, \mathscr{K}] \subset \mathscr{K} \qquad [\mathscr{K}, \mathscr{K}] \subset \mathscr{H}.$$

Usually, the σ-field on the symmetric space is expressed as

$$N(x) = g(x) n g^{-1}(x)$$

where $g(x)$ is the group element of $SO(N)$. Obviously,

$$N(x)^2 = 1. \tag{6}$$

Then the Lagrangian has the following form:

$$\mathscr{L}(x) = \tfrac{1}{16}\,\mathrm{Tr}(\partial_\mu N(x)\,\partial^\mu N(x)). \tag{7}$$

Varying $\mathscr{L}(x)$ under the constraint condition (6), we obtain the motion equation

$$\partial_\mu K^\mu(x) = 0 \tag{8}$$

where

$$K_\mu(x) = -\tfrac{1}{2} N(x)\,\partial_\mu N(x). \tag{9}$$

The conserved Noether currents are

$$j_\mu(x) = -K_\mu(x).$$

According to (5), the left-invariant Maurer–Cartan form $a_\mu(x)$ also has a decomposition:

$$a_\mu(x) \equiv g^{-1}(x)\,\partial_\mu g(x) = h_\mu(x) + k_\mu(x) \tag{10}$$

where

$$h_\mu(x) = \tfrac{1}{2}[a_\mu, n]_+ n \in \mathscr{H}$$
$$k_\mu(x) = \tfrac{1}{2}[a_\mu, n]n = g^{-1}(x)K_\mu(x)g(x) \in \mathscr{K}. \tag{11}$$

From (7), (9) and (11), we get

$$\mathscr{L} = -\tfrac{1}{2}(k_\mu(x), k^\mu(x)) \tag{12}$$

where (,) is the G-invariant inner product on the coset space, induced from the Killing–Cartan form of the Lie algebra $\mathscr{G}$. Correspondingly, the motion equation (8) can be expressed as

$$D_\mu k^\mu \equiv \partial_\mu k^\mu + [h_\mu, k^\mu] = 0. \tag{13}$$

On the other hand, the pure gauge potential $a_\mu(x)$ satisfies the Maurer–Cartan equations:

$$\partial_\mu h_\nu - \partial_\nu h_\mu + [h_\mu, h_\nu] + [k_\mu, k_\nu] = 0 \tag{14}$$

$$D_\mu k_\nu - D_\nu k_\mu = 0. \tag{15}$$

Let

$$^*k_\mu = \varepsilon_{\mu\nu} k^\nu \qquad (-\varepsilon_{01} = \varepsilon_{10} = 1)$$

then (15) becomes

$$D_\mu^* k^\mu(x) = 0. \tag{16}$$

Comparing with (13), we see that the theory admits a continual dual transformation. The result allows us to introduce a real linear combination of $k^\mu(x)$ and $^*k^\mu(x)$

$$\tilde{k}_\mu(x, \lambda) = \mathrm{ch}\,\phi k_\mu(x) + \mathrm{sh}\,\phi^* k_\mu(x)$$

where

$$\operatorname{ch}\phi=\frac{\lambda^2+1}{\lambda^2-1} \qquad \operatorname{sh}\phi=\frac{2\lambda}{\lambda^2-1}.$$

Then $h_\mu(x)$ and $\tilde{k}_\mu(x,\lambda)$ satisfy the same equations as $h_\mu(x)$ and $k_\mu(x)$:

$$\partial_\mu h_\nu-\partial_\nu h_\mu+[h_\mu,h_\nu]+[\tilde{k}_\mu,\tilde{k}_\nu]=0 \tag{17}$$

$$D_\mu\tilde{k}^\mu(x,\lambda)=0. \tag{18}$$

It means that $h_\mu(x)+\tilde{k}_\mu(x,\lambda)$ can also be expressed as a pure gauge, namely,

$$\partial_\mu\Phi(x,\lambda)=\Phi(x,\lambda)(h_\mu(x)+\tilde{k}_\mu(x,\lambda)) \tag{19}$$

$$\Phi(x,0)=g^{-1}(x).$$

We take these as the Lax pair equations in moving frames. The spatial part of Lax matrices is

$$L(x,\lambda)=h_1(x)+\operatorname{ch}\phi k_1(x)+\operatorname{sh}\phi k_0(x). \tag{20}$$

Usually, one constructs another auxiliary linear equation

$$\partial_\mu U(x,\lambda)=U(x,\lambda)\frac{2}{1-\lambda^2}(j_\mu+\lambda\varepsilon_{\mu\nu}j^\nu) \tag{21}$$

whose spatial part of Lax matrices is

$$L(x,\lambda)=\frac{2}{1-\lambda^2}(j_1(x)+\lambda j_0(x)). \tag{22}$$

According to Maillet [1], the Poisson bracket between the Lax potential should be

$$\begin{aligned}\{L(x,\lambda)\otimes L(y,\mu)\}=&-[r(x,\lambda,\mu),L(x,\lambda)\otimes 1+1\otimes L(x,\mu)]\delta(x-y)\\ &+[s(x,\lambda,\mu),L(x,\lambda)\otimes 1-1\otimes L(x,\mu)]\delta(x-y)\\ &-(r(x,\lambda,\mu)+s(x,\lambda,\mu)-r(y,\lambda,\mu)+s(y,\lambda,\mu))\delta'(x-y).\end{aligned} \tag{3}$$

Using (22), Forger *et al* have given r- and s-matrices as [2]

$$r(x,\lambda,\mu)=-\frac{2\lambda\mu}{(1-\lambda\mu)(\lambda-\mu)}C-\frac{2(1+\lambda\mu)(\lambda-\mu)}{(1-\lambda\mu)(1-\lambda^2)(1-\mu^2)}j(x) \tag{23}$$

$$s(x,\lambda,\mu)=-\frac{2(\lambda+\mu)}{(1-\lambda^2)(1-\mu^2)}j(x) \tag{24}$$

where C is the Casimir tensor and $j(x)$ is a scalar field.

In the next section we will calculate the r- and s-matrices for the O(N) σ-model by using the local moving-frame method, namely, we will take equation (20) rather than (22) as our starting point. The reason is that we can gauge transform (20) conveniently and see how the r- and s-matrices change under gauge transformation. Thus the geometrical characteristics of the r- and s-matrices can clearly be seen.

3. The r- and s-matrices in the moving frame

The group element g of $SO(N)$ can be written as

$$g=g'h$$

where $h\in SO(N-1)$ and $g'\in SO(N)/SO(N-1)$. For simplicity, first we take the Schwinger gauge, $h=1$, namely, $g=g'$. Now we can choose g as [3]

$$g=R_1(\theta_1)R_2(\theta_2)\ldots R_{N-1}(\theta_{N-1}) \tag{25}$$

where $R_i(\theta)=\exp(\theta T^{i(i+1)})$ and the generators T^{ab} of $SO(N)$ can be chosen as

$$(T^{ab})_{cd}=\delta_{ac}\delta_{bd}-\delta_{bc}\delta_{ad}.$$

Their commutation relations are

$$[T^{ab},T^{cd}]=\delta_{ad}T^{bc}+\delta_{bc}T^{ad}-\delta_{ac}T^{bd}-\delta_{bd}T^{ac}.$$

By some calculation, we get

$$g^{-1}\,\mathrm{d}g=\sum_{i=1}^{N-2}\mathrm{d}\theta_i\sum_{j=i+1}^{N-1}T^{ij}s_{i+1}\ldots s_{j-1}c_j+\sum_{i=1}^{N-1}\mathrm{d}\theta_i\,T^is_{i+1}s_{i+2}\ldots s_{N-1} \tag{26}$$

where $s_i\equiv\sin\theta_i$, $c_i\equiv\cos\theta_i$ and $T^i\equiv T^{iN}$.

If we set diagonal matrix $n=\{1,1,\ldots,1,-1\}$, then $T^{ij}\in\mathscr{H}$, $T^i\in\mathscr{K}$. According to (10), it is easy to get h_μ, k_μ as

$$h_\mu=\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}(\partial_\mu\theta_i)T^{ij}s_{i+1}\ldots s_{j-1}c_j \tag{27}$$

$$k_\mu=\sum_{i=1}^{N-1}(\partial_\mu\theta_i)T^is_{i+1}s_{i+2}\ldots s_{N-1}. \tag{28}$$

Then from (12), the expression for the Lagrangian is

$$\mathscr{L}=\sum_{i=1}^{N-1}(\partial^\mu\theta_i\,\partial_\mu\theta_i)s_{i+1}^2s_{i+2}^2\ldots s_{N-1}^2. \tag{29}$$

Consequently, the canonical momenta π_i have the following form

$$\pi_i=2\frac{\mathrm{d}\theta_i}{\mathrm{d}t}s_{i+1}^2s_{i+2}^2\ldots s_{N-1}^2. \tag{30}$$

The fundamental Poisson brackets are:

$$\begin{aligned}&\{\theta_i(x),\pi_j(y)\}=\delta_{ij}\delta(x-y)\\&\{\theta_i(x),\theta_j(y)\}=\{\pi_i(x),\pi_j(y)\}=0.\end{aligned} \tag{31}$$

Using the above formulae and the following notations:

$$\Gamma^i(x)=\sum_{j=i+1}^{N-1}\Theta_i(x)T^{ij}\qquad\Gamma^{N-1}=0$$

$$\Theta_i(x)=\frac{c_j}{s_js_{j+1}\ldots s_{N-1}}$$

$$J_k=\frac{1}{2}\sum_{i=1}^{N-1}T^i\otimes T^i\qquad J_h=\frac{1}{2}\sum_{i=1}^{N-2}\sum_{j=i+1}^{N-1}T_{ij}\otimes T_{ij}$$

$$P_1(x)=\frac{1}{2}\sum_{i=1}^{N-1} T^i\otimes\Gamma^i(x) \qquad P_2(x)=\frac{1}{2}\sum_{i=1}^{N-1}\Gamma^i(x)\otimes T^i$$

we begin to calculate the Poisson brackets between the Lax potential (20).

Take

$$L(x,\lambda)=h_1(x)+\operatorname{ch}\phi_1 k_1(x)+\operatorname{sh}\phi_1 k_0(x)$$

$$L(x,\mu)=h_1(x)+\operatorname{ch}\phi_2 k_1(x)+\operatorname{sh}\phi_2 k_0(x)$$

then

$$\begin{aligned}
\{L(x,\lambda)\overset{\otimes}{,}L(y,\mu)\}
&=\operatorname{sh}\phi_2\{h_1(x)\overset{\otimes}{,}k_0(y)\}+\operatorname{ch}\phi_1\operatorname{sh}\phi_2\{k_1(x)\overset{\otimes}{,}k_0(y)\}+\operatorname{sh}\phi_1\{k_0(x)\overset{\otimes}{,}h_1(y)\}\\
&\quad+\operatorname{sh}\phi_1\operatorname{ch}\phi_2\{k_0(x)\overset{\otimes}{,}k_1(y)\}+\operatorname{sh}\phi_1\operatorname{sh}\phi_2\{k_0(x)\overset{\otimes}{,}k_0(y)\}\\
&=-[(\operatorname{sh}\phi_2 J_k+\operatorname{ch}\phi_1\operatorname{sh}\phi_2 P_2(x)),k_1(x)\otimes 1]\delta(x-y)\\
&\quad+[\operatorname{sh}\phi_1 J_k+\operatorname{sh}\phi_1\operatorname{ch}\phi_2 P_1(x),1\otimes k_1(x)]\delta(x-y)\\
&\quad-[(\operatorname{sh}\phi_2 P_2+\operatorname{ch}\phi_1\operatorname{sh}\phi_2 J_k,h_1(x)\otimes 1]\delta(x-y)\\
&\quad+[(\operatorname{sh}\phi_1 P_2(x)+\operatorname{sh}\phi_1\operatorname{ch}\phi_2 J_k),1\otimes h_1(x)]\delta(x-y)\\
&\quad+[\operatorname{sh}\phi_1\operatorname{sh}\phi_2 P_1(x),1\otimes k_0(x)]\delta(x-y)\\
&\quad-[\operatorname{sh}\phi_1\operatorname{sh}\phi_2 P_2(x),k_0(x)\otimes 1]\delta(x-y)\\
&\quad+(\operatorname{sh}\phi_1 P_1(x)+\operatorname{sh}\phi_2 P_2(y)+\operatorname{ch}\phi_1\operatorname{sh}\phi_2 J_k+\operatorname{ch}\phi_2\operatorname{sh}\phi_1 J_k)\delta'(x-y).
\end{aligned}\tag{32}$$

Comparing with equation (3), we immediately get the matrix $s(x,\lambda,\mu)$:

$$s(x,\lambda,\mu)=-\tfrac{1}{2}\operatorname{sh}(\phi_1+\phi_2)J_k-\tfrac{1}{2}\operatorname{sh}\phi_1 P_1(x)-\tfrac{1}{2}\operatorname{sh}\phi_2 P_2(x). \tag{33}$$

Then assuming

$$r(x,\lambda,\mu)=\tfrac{1}{2}AJ_k+\tfrac{1}{2}BJ_h-\tfrac{1}{2}\operatorname{sh}\phi_1 P_1(x)+\tfrac{1}{2}\operatorname{sh}\phi_2 P_2(x)$$

and using the following identities:

$$[J_k,k_\mu\otimes 1]+[J_h,1\otimes k_\mu]=0$$

$$[J_h,k_\mu\otimes 1]+[J_k,1\otimes k_\mu]=0$$

$$[J_k,h_\mu\otimes 1+1\otimes h_\mu]=0$$

$$[J_h,h_\mu\otimes 1+1\otimes h_\mu]=0$$

we also get the matrix $r(x,\lambda,\mu)$

$$r(x,\lambda,\mu)=-\frac{\operatorname{sh}^2\phi_1+\operatorname{sh}^2\phi_2}{2\operatorname{sh}(\phi_1-\phi_2)}J_k-\frac{\operatorname{sh}\phi_1\operatorname{sh}\phi_2}{\operatorname{sh}(\phi_1-\phi_2)}J_h-\tfrac{1}{2}\operatorname{sh}\phi_1 P_1(x)+\tfrac{1}{2}\operatorname{sh}\phi_2 P_2(x). \tag{34}$$

Here we see that the field-dependent terms of the r- and s-matrices are only related to $\Theta_i(x)$, the Riemannian connection under the Schwinger gauge on S^{N-1} [3], which can be seen more clearly under the gauge transformation given in the next section.

4. Gauge transformation

Now let's take a look at how r and s change under gauge transformation. After a gauge transformation h, the following changes take place

$$h_\mu(x) \to h'_\mu(x) = h^{-1}(x) h_\mu(x) h(x) + h^{-1}(x)\, \partial_\mu h(x) \tag{35}$$

$$k_\mu(x) \to k'_\mu(x) = h^{-1}(x) k_\mu(x) h(x) \tag{36}$$

$$\begin{aligned} L(x, \lambda) &\to L'(x, \lambda) \\ &= h'_1(x) + \operatorname{ch}\phi k'_1(x) + \operatorname{sh}\phi k'_0(x) = h^{-1}(x) h_1(x) h(x) + \operatorname{ch}\phi h^{-1}(x) k_1(x) h(x) \\ &\quad + \operatorname{sh}\phi h^{-1}(x) k_0(x) h(x) + h_{-1}(x)\, \partial_1 h(x). \end{aligned} \tag{37}$$

Noting the identity

$$(f(x) - f(y))\delta'(x-y) = -f'(x)\delta(x-y).$$

we find the changes below:

$$\begin{aligned} r(x, \lambda, \mu)\delta(x-y) &\to r'(x, \lambda, \mu)\delta(x-y) \\ &= h^{-1}(x) \otimes h^{-1}(y)[r(x, \lambda, \mu)\delta(x-y) - \tfrac{1}{2}(1 \otimes h(y)\{L(x, \lambda) \underset{,}{\otimes} h^{-1}(y)\} \\ &\quad - h(x) \otimes 1\{h^{-1}(x) \underset{,}{\otimes} L(y, \mu)\})] h(x) \otimes h(y) \end{aligned}$$

$$\begin{aligned} s(x, \lambda, \mu)\delta(x-y) &\to s'(x, \lambda, \mu)\delta(x-y) \\ &= h^{-1}(x) \otimes h^{-1}(y)[s(x, \lambda, \mu)\delta(x-y) - \tfrac{1}{2}(1 \otimes h(y)\{L(x, \lambda) \underset{,}{\otimes} h^{-1}(y)\} \\ &\quad + h(x) \otimes 1\{h^{-1}(x) \underset{,}{\otimes} L(y, \mu)\})] h(x) \otimes h(y). \end{aligned} \tag{39}$$

Since

$$1 \otimes h(y)\{k_0(x) \underset{,}{\otimes} h^{-1}(y)\} = \left[\sum_{i=1}^{N-1} \frac{-1}{2k_i} T^i \otimes (h\partial_i h^{-1})\right] \delta(x-y) \tag{40}$$

$$h(x) \otimes 1\{h^{-1}(x) \underset{,}{\otimes} k_0(y)\} = \left[\sum_{i=1}^{N-1} \frac{-1}{2k_i} (h\partial_i h^{-1}) \otimes T^i\right] \delta(x-y) \tag{41}$$

where

$$k_i = s_{i+1} s_{i+2} \ldots s_{N-1}$$

eventually we get

$$r'(x, \lambda, \mu) = -\frac{\operatorname{sh}^2\phi_1 + \operatorname{sh}^2\phi_2}{2\operatorname{sh}(\phi_1 - \phi_2)} J_k - \frac{\operatorname{sh}\phi_1 \operatorname{sh}\phi_2}{\operatorname{sh}(\phi_1 - \phi_2)} J_h - \tfrac{1}{2}\operatorname{sh}\phi_1 P'_1(x) + \tfrac{1}{2}\operatorname{sh}\phi_2 P'_2(x) \tag{42}$$

$$s'(x, \lambda, \mu) = -\tfrac{1}{2}\operatorname{sh}(\phi_1 + \phi_2) J_k - \tfrac{1}{2}\operatorname{sh}\phi_1 P'_1(x) - \tfrac{1}{2}\operatorname{sh}\phi_2 P'_2(x) \tag{43}$$

where

$$P_1^{\prime}=\frac{1}{2}\sum_{i=1}^{N-1}(h^{-1}T^ih)\otimes\Gamma_h^i$$

$$P_2^{\prime}=\frac{1}{2}\sum_{i=1}^{N-1}\Gamma_h^i\otimes(h^{-1}T^ih)$$

$$\Gamma_h^i=\frac{h^{-1}\partial_ih}{k_i}+h^{-1}\Gamma^ih. \tag{44}$$

From equation (44) we can see that Γ^i is just a Riemmanian connection matrix on S^{N-1}, since the way in which it changes under a gauge transformation is the same as a connection. For example, if we take

$$h(x)=R_{N-2}(\mp\theta_{N-2})\ldots R_2(-\theta_2)R_1(-\theta_1) \tag{45}$$

from (44) we obtain

$$\Gamma_h^i=\frac{\mp1+\cos\theta_{N-1}}{\sin\theta_{N-1}}(h^{-1}T^{i(N-1)}h) \tag{46}$$

which is exactly the Riemannian connection under the Wu–Yang gauge [3].

In order to relate our r- and s-matrices to the r- and s-matrices given by Maillet and Forger *et al*, we take another special gauge transformation by replacing h^{-1} with g. Then there exists

$$H_\mu=gh_\mu g^{-1}+g\,\partial_\mu g^{-1}=j_\mu$$

$$K_\mu=gk_\mu g^{-1}=-j_\mu.$$

Putting these two formulas into (19), we get the common linear equation (21). Moreover, noticing

$$1\otimes g^{-1}(y)\}k_0(x)\overset{,}{\otimes}g(y)\}=-(J_k+P_1^{\prime}x))\delta(x-y)$$

$$g^{-1}(x)\otimes 1\{g(x)\underset{,}{\otimes}k_0(y)\}=(J_k+P_2^{\prime}(x))\delta(x-y)$$

and replacing sh ϕ_1, ch ϕ_1 and sh ϕ_2, ch ϕ_2 with λ, μ respectively, we also get equations (23) and (24). So the two different forms of r- and s-matrices can be associated by a special gauge transformation, or a frame change, but our r- and s-matrices have more clear geometric meaning: the field-dependent terms are only related to the Riemmanian connection on the target manifold S^{N-1}.

References

[1] Maillet J M 1985 *Phys. Lett.* **162B** 137
Maillet J M 1986 *Nucl. Phys.* B **269** 54–76
[2] Forger M, Larry J and Schaper U 1992 *Commun. Math. Phys.* **146** 397
Bordemann M, Forger M, Larry J and Schaper U 1993 *Commun. Math. Phys.* **152** 167
[3] Bo-yu Hou, Bo-yuan Hou and Pei Wang 1982 *J. Math. Phys.* **23** 2488
[4] Bo-yu Hou, Bo-yuan Hou and Pei Wang 1985 *J. Phys. A: Math. Gen.* **18** 165
Bo-yu Hou 1984 *J. Math. Phys.* **25** 2325
[5] Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in Solitons* (Berlin: Springer)